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LETTER TO THE EDITOR

A generalisation of a solvable model in population dynamics

George DeRise† and John A Adam‡

† Division of Natural Science and Mathematics, Thomas Nelson Community College, Hampton, VA 23670, USA

‡ Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA 23529-0077, USA

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Abstract. Biswas and Karmakar were the first to consider a periodic rate transform for an autocatalytic growth process, $G(x)=-\tan(\alpha \ln x)$. They obtained an exact analytic solution for the probability density function by solving the equivalent Schrödinger equation for the Fokker–Planck equation, and utilised this function to calculate various moments. In this letter we extend their work to a more general rate transform $G(x)=\alpha\tan(\beta \ln x)+\delta\cot(\beta\ln x)$.

The growth of a population with random environmental influences has been considered by Goel *et al* [1] and Montroll [2]. They assumed that the population $N(t)$ satisfied a stochastic differential equation of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t}=kNG\left(\frac{N}{\theta}\right)+NF(t) \tag{1}$$

where k is a (constant) growth rate parameter, G is a growth rate function, θ is the (constant) saturation level and $F(t)$ is a random function representing chance, unspecified influences not taken care of by G. $F(t)$ has the properties

(1) $\langle F(t)\rangle=0$

(2) $\langle F(t)\cdot F(t')\rangle=\sigma^2\delta(t-t')$

(3) $F(t)$ is a Gaussian process

i.e. $F(t)$ represents Gaussian white noise.

Working with the more convenient variable $v=\ln N/\theta$ (1) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}t}=kG(\mathrm{e}^v)+F(t). \tag{2}$$

By a stochastic argument Goel *et al* derive a Fokker–Planck equation for the probability $P(v,t)$, that $\log N/\theta$ has a value v at time t,

$$\frac{\partial P}{\partial t}=-k\frac{\partial}{\partial v}\{PG(\mathrm{e}^v)\}+\tfrac{1}{2}\sigma^2\frac{\partial^2 P}{\partial v^2}. \tag{3}$$

If we set

$$P=\Psi(v,t)\exp\left\{\frac{k}{\sigma^2}\int_a^v G(\mathrm{e}^v)\,\mathrm{d}v\right\} \tag{4}$$

(3) becomes the 'equivalent' Schrödinger equation

$$\frac{2}{k}\frac{\partial\Psi}{\partial t}=\frac{\sigma^2}{k}\frac{\partial^2\Psi}{\partial v^2}-\left\{\frac{k}{\sigma^2}[G(\mathrm{e}^v)]^2+\frac{\partial}{\partial v}G(\mathrm{e}^v)\right\}\Psi. \tag{5}$$

Separating variables by letting $\Psi(v,t)=\mathrm{e}^{-\hat{a}t}X(v)$, the spatial dependence reduces to the Schrödinger equation

$$\frac{\mathrm{d}^2X}{\mathrm{d}v^2}+\left(\frac{kE}{\sigma^2}-U\right)X=0 \tag{6}$$

where $-E$ is the separation constant, and U is the Schrödinger potential given by the expression in braces in (5). If

$$G(\mathrm{e}^v)=\alpha\tan\beta v+\delta\cot\beta v$$

then letting $\xi=\beta v$ in (6) and defining

$$E=\frac{k}{\beta^2\sigma^2}\left(\frac{2\hat{a}}{k}-\frac{2\alpha k\delta}{\sigma^2}+\frac{k\alpha^2}{\sigma^2}+\frac{k\delta^2}{\sigma^2}\right) \tag{7}$$

$$\frac{k}{\beta^2\sigma^2}\left(\frac{k\alpha^2}{\sigma^2}+\alpha\beta\right)=\lambda(\lambda-1)\qquad\text{and}\qquad\frac{k}{\beta^2\sigma^2}\left(\frac{k\delta^2}{\sigma^2}-\delta\beta\right)=\kappa(\kappa-1) \tag{8}$$

(6) becomes

$$\frac{\mathrm{d}^2X}{\mathrm{d}\xi^2}+\left[E-\left(\frac{\lambda(\lambda-1)}{\cos^2\xi}+\frac{\kappa(\kappa-1)}{\sin^2\xi}\right)\right]X=0\qquad\lambda,\kappa>1. \tag{9}$$

Choosing the single branch $0\leqslant\xi\leqslant\pi/2$ we solve (9) with boundary conditions $X=0$ for $\xi=0$, $X=0$ for $\xi=\pi/2$. This is a classic problem in quantum mechanics, namely that of the solution of the Schrödinger equation for the first Pöschl–Teller potential [3].

First transform to the new independent variable $y=\sin^2\xi$ obtaining

$$y(1-y)X''(y)+(\tfrac{1}{2}-y)X'(y)+\tfrac{1}{4}\left[E-\left(\frac{\lambda(\lambda-1)}{(1-y)}+\frac{\kappa(\kappa-1)}{y}\right)\right]X=0 \tag{10}$$

and then to the new dependent variable

$$X=y^\mu(1-y)^\gamma f(y) \tag{11}$$

we get

$$y(1-y)f''(y)+[(\kappa+\tfrac{1}{2})-(\kappa+\lambda+1)y]f'(y)+\tfrac{1}{4}[E-(\kappa+\lambda)^2]f=0 \tag{12}$$

by choosing $\mu=\kappa/2$, $\gamma=\lambda/2$. Equation (12) is a hypergeometric differential equation which has a general solution [4]

$$f=C_1\,{}_2F_1(a,b;c;y)+C_2y^{1-c}\,{}_2F_1(a+1-c,b+1-c;2-c;y) \tag{13}$$

where

$$a=\tfrac{1}{2}(\kappa+\lambda\pm\sqrt{E})\qquad b=\tfrac{1}{2}(\kappa+\lambda\mp\sqrt{E})\qquad c=\kappa+\tfrac{1}{2} \tag{14}$$

since $X=0$ for $y=0$, $C_2=0$. To obtain the condition at $y=1$ we use the well known transformation 15.3.6 in [5]. In the vicinity of $y=1$ we must take $b=-n$, $n=0,1,2,\ldots$ to avoid the singularity due to the negative exponent $\frac{1}{2}-\lambda$ in the second term. Using

(14) we find $a = \kappa + \lambda + n$ and the eigenvalues $E_n = (\kappa + \lambda + 2n)^2$. The corresponding eigenfunctions are

$$X_n = \sin^\kappa \xi \cos^\lambda \xi \, {}_2F_1(\kappa + \lambda + n, -n; \kappa + \tfrac{1}{2}; \sin^2 \xi). \tag{15}$$

Since the Jacobi polynomials are defined by [6]

$$J_n(p, q; x) = {}_2F_1(p + n, -n; q; x) \tag{16}$$

the eigenfunctions X_n can be written

$$X_n = \sin^\kappa \xi \cos^\lambda \xi J_n(\kappa + \lambda, \kappa + \tfrac{1}{2}; \sin^2 \xi). \tag{17}$$

The Jacobi polynomials have the orthogonality condition [5]

$$\int_0^1 x^{q-1}(1-x)^{p-q} J_n(p, q; x) J_m(p, q; x) \, \mathrm{d}x = N(p, q, n)\delta_{mn} \qquad q > 0, p - q > -1 \tag{18}$$

where the explicit form of N is of no consequence for our discussion.

Thus,

$$\int_0^{\pi/2} \sin^{2\kappa} \xi \cos^{2\lambda} \xi J_n(\kappa + \lambda, \kappa + \tfrac{1}{2}; \sin^2 \xi) J_m(\kappa + \lambda, \kappa + \tfrac{1}{2}; \sin^2 \xi) \, \mathrm{d}\xi = \frac{N\delta_{mn}}{2} \qquad \kappa > -\tfrac{1}{2}, \lambda > -\tfrac{1}{2}. \tag{19}$$

The most general solution for Ψ is

$$\Psi(v, t) = \sum_{n=0}^{\infty} C_n X_n(\xi) \, \mathrm{e}^{-\hat{a}t} \tag{20}$$

where $\hat{a}$ is easily found from our expressions for E and e_n.

Using (4)

$$P(v, t) = \Psi(v, t) \bar{K} \sec^{\alpha\beta} \beta v \sin^{\delta\beta} \beta v \tag{21}$$

with appropriate conditions on $\alpha\beta$, $\delta\beta$ from (8) where $\bar{K}$ is the constant value of the integrated expression at the lower limit of integration.

From (21),

$$\Psi(v, 0) = \frac{1}{\bar{K}} \cos^{\alpha\beta} \xi \operatorname{cosec}^{\delta\beta} \xi P(v, 0). \tag{22}$$

From (20),

$$\int_0^{\pi/2} \Psi(v, 0) X_m(\xi) \, \mathrm{d}\xi = \sum_{n=0}^{\infty} C_n \int_0^{\pi/2} X_n(\xi) X_m(\xi) \, \mathrm{d}\xi. \tag{23}$$

Now utilising (17), (19) and (22) we get

$$\int_0^{\pi/2} \frac{1}{\bar{K}} P(v, 0) \cos^{\alpha\beta} \xi \operatorname{cosec}^{\delta\beta} \xi X_m(\xi) \, \mathrm{d}\xi = \tfrac{1}{2} C_n N(\kappa, \lambda, n). \tag{24}$$

Set $P(v, 0) = \delta(v - v_0)$ with $\xi_0 = v_0\beta$ corresponding to an initial spiked distribution and we obtain

$$C_n = \left(\frac{2}{N}\right) \cos^{\alpha\beta} \xi_0 \operatorname{cosec}^{\delta\beta} \xi_0 X_m(\xi_0). \tag{25}$$

Now using (17), (20), (21) and (25) we derive an expression for the probability density function $P(v, t)$

$$P(v, t) = \sum_{n=0}^{\infty} \frac{2}{N} \sin^{\kappa+\delta\beta} \xi \cos^{\lambda-\alpha\beta} \xi \sin^{\kappa-\delta\beta} \xi_0 \cos^{\lambda+\alpha\beta} \xi_0 J_m(\xi_0) J_n(\xi) \, \mathrm{e}^{-\hat{a}t}. \tag{26}$$

This reduces to the analogous expression given by Biswas and Karmakar [7] when the appropriate substitutions are made. Choosing $v_0\beta = \pi/4$ so that all terms involving ξ_0 are incorporated into a constant M

$$P(v, t) = \sum_{n=0}^{\infty} M \sin^{\kappa+\delta\beta} \xi \cos^{\lambda-\alpha\beta} \xi J_n(\xi) \, \mathrm{e}^{-\hat{a}t}. \tag{27}$$

The moments of N/θ can easily be found from this density function

$$\begin{aligned}\left\langle \left(\frac{N}{\theta}\right)^{2\tilde{\lambda}} \right\rangle &= \langle \mathrm{e}^{2\tilde{\lambda} v} \rangle \\ &= \int_0^{\pi\beta/2} P(v, t) \, \mathrm{e}^{2\tilde{\lambda} v} \, \mathrm{d}v \\ &= \int_9^{\pi/2} \sum_{n=0}^{\infty} \frac{M}{\beta} \sin^{\kappa+\delta\beta} \xi \cos^{\lambda-\alpha\beta} \xi J_n(\kappa+\lambda, \kappa+\tfrac{1}{2}; \sin^2 \xi) \\ &\quad \times \exp\left(\frac{2\tilde{\lambda}\xi}{\beta} - \hat{a}t\right) \mathrm{d}\xi. \end{aligned} \tag{28}$$

The Jacobi polynomials may be written as

$$J_n(p, q, \varphi) = 1 + \sum_{l-1}^{n} f(l, n) \varphi^l \tag{29}$$

where

$$f(l, n) = (-1)^l \binom{n}{l} \frac{(p+n)(p+n+1)\ldots(p+n+l-1)}{q(q+1)\ldots(q+l-1)} \tag{30}$$

so that (28) becomes

$$\left\langle \left(\frac{N}{\theta}\right)^{2\tilde{\lambda}} \right\rangle = \sum_{n=0}^{\infty} \frac{M}{\beta} \mathrm{e}^{-\hat{a}t} \left\{ I_1(\alpha, \beta, \delta, \kappa, \lambda, \tilde{\lambda}, \xi) + \sum_{l=1}^{n} f(l, n) I_2(\alpha, \beta, \delta, \kappa, \lambda, l, \tilde{\lambda}, \xi) \right\} \tag{31}$$

where

$$I_1 = \int_0^{\pi/2} \sin^{\kappa+\delta\beta} \xi \cos^{\lambda-\alpha\beta} \xi \, \mathrm{e}^{2\tilde{\lambda}\xi/\beta} \, \mathrm{d}\xi \tag{32}$$

$$I_2 = \int_0^{\pi/2} \sin^{\kappa+\delta\beta+2l} \xi \cos^{\lambda-\alpha\beta} \xi \, \mathrm{e}^{2\tilde{\lambda}\xi/\beta} \, \mathrm{d}\xi. \tag{33}$$

The various moments can be calculated by letting $\tilde{\lambda} = \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

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